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PROBABILISTIC NETWORK STUDY—SYSTEM RELIABILITY

BY

F. M. Reza

ELECTRICAL ENGINEERING DEPARTMENT

COLLEGE OF ENGINEERING

SYRACUSE UNIVERSITY RESEARCH INSTITUTE

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F. M. Reza

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The research reported here as well as in the three previously distributed scientific reports was performed by the project director, Dr. F. M. Reza, and several graduate students at Syracuse University. Acknowledgment is made for the sponsorship of the Air Force Cambridge Research Laboratories, Laurence G. Hanscom Field, Bedford, Massachusetts authorized under contract AF 19(604)-6169.

ABSTRACT OF THE PRESENT REPORT

Part I gives a formal proof for a well-known statement of Shannon. Shannon states that for very large T, the number of distinct permissible messages of length T behaves as $A_i \, r_i^T$, where r_i is the largest positive root of the characteristic equation. It is proved here that the characteristic equation cannot possess complex conjugate roots leading to a dominant term.

Part II presents a note on a new sampling theorem using Bernstein's orthogonal polynomials as the basis. Some similarity between the problem of approximation with $B_n(t)$ and the fundamental theorem of information theory is denoted.

Part III contains a new relation between positive real functions of Circuit Theory and the h(p) functions of Reliability Theory. A mathematical transformation is described for associating a realizable impedance function to a realizable realiability function. This idea provides a new source of constraints for realizable reliability functions.

Part IV contains merely the formulation of a new problem concerning "nearly normal channels" and the evaluation of their information transmission.

ABSTRACT OF AFCRC-TN-59-588---BASIC CONCEPTS OF INFORMATION THEORY FINITE SCHEME

The report presents basic concepts of information theory in the light of set theory and probability. The derivation of the functional form of the entropy function is given and its properties are discussed in detail. The report also gives a set theory interpretation of Shannon's Fundamental Inequalities, for entropies of more complex systems. The concept of channel capacity is discussed in detail with examples of BSC and BEC.

ABSTRACT OF AFCRC-TR-60-126---STUDY OF SYSTEM RELIABILITY

This report discusses continuous channels without memory. It describes a measure of information in the continuous channels and the maximization of the entropy under reasonable circumstances. In particular, the transmission of information in presence of additive noise is formulated and Shannon's well-known formula for Gaussian formula channels derived.

ABSTRACT OF AFCRL 503--THREE PROOFS FOR THE FUNDAMENTAL THEOREM OF INFORMATION THEORY DISCRETE MEMORYLESS CHANNELS

This report considers the fundamental theorem of information theory for discrete memoryless channels. Subsequent to some preliminary preparations, describing the decision scheme and average error probability, three proofs of the basic theorem are presented

Feinstein's Proof,

Shannon's Proof,

Wolfowitz's Proof.

A discussion of the bounds of error probability and their relations to the word length and the converse of the fundamental theorem is included.

SUMMARY AND ANALYSIS OF RESEARCH UNDER THE CONTRACT

The project began with the study of the reliability in communication systems. In retrospect, it can be seen that the research under the contract has assumed two parallel aspects, one based on probabilistic studies and the other based on information theory ideas.

Information Theory predicts the possibility of the most reliable transmission of information. In this context, our studies were directed toward a clear understanding of basic concepts of information theory. Subsequent to such studies one is enabled to clearly analyse the rate of the transmission of information of simple channels, the associated errors and reliability (Technical Reports 1 and 3).

In connection with Probabilistic Networks per se, we have analyzed some of the existing literature and promoted a new question. Namely, what are the necessary and sufficient conditions for an $h(p)$ function to represent the reliability of a realizable network. Several Master's and Doctoral candidates were guided in this area. Although, they have produced a number of interesting results, this particular problem still remains unsolved. Some discussion of interest is presented in Part III of this final report.

As a junction of the above two directions we came across a new form of a sampling theorem with potential applications to both of these areas. This theorem and its ramifications are discussed in Part II of this report.

As far as the further future work in this area is concerned, the analysis and synthesis of probabilistic networks requires far more attention. Also, the study of nearly gaussian channels and their reliability warrants special investigation.

Part I

A NOTE ON THE CAPACITY OF DISCRETE NOISELESS CHANNELS

At the very beginning of his well-known paper, Shannon considers the transmission of information through a memoryless discrete noiseless channel. A memoryless source selects symbols, at random from a specified alphabet

$$[a_1, a_2, \ldots, a_n]$$

Each transmitted symbol goes through a noiseless channel without alteration, but has a duration or cost factor assigned to it. The costs of different symbols are prespecified by a matrix such as

$$[t_1, t_2, \ldots, t_n]$$

Sequences of symbols (called message) are selected by the source with some possible restrictions. We may specify for instance that words of the type $a_1 a_1$ are not to be transmitted. For example, in the ordinary telegraphy, the source is required to omit repetition of spaces. The question exposed and answered by Shannon is how one can measure the capacity of such a channel. Shannon defines the capacity of this channel as

$$C = \lim_{T \to \infty} \frac{\log N(T)}{T}$$

where $N(T)$ is the number of permissible messages of total duration or cost T. This maximum rate of the transmission of information may be only obtained when all the distinct messages are equiprobable.

The evaluation of $N(T)$ requires the solution of the following well-known difference equation[1,2]

$$N(T) = \sum_{k=1}^{n} N(T - t_k)$$

When all symbol costs are distinct, then the solution to this difference equation is

$$N(T) = \sum_{k=1}^{n} A_k r_k^T$$

where A_k's depend on the boundary values of the problem and r_k's are the roots (distinct) of the characteristic equation

$$1 = \sum_{k=1}^{n} r^{-t_k}$$

When several symbols may be of the same duration, then we may conveniently rearrange the order of appearance of the symbols in the cost matrix to read:

$$[\ [t_1],\ [t_2],\ .\ .\ .,\ [t_m]\ ]$$

$$[\ \alpha_1,\ \alpha_2,\ .\ .\ .,\ \alpha_m]$$

$$t_1 < t_2 < \ .\ .\ .\ < t_m \qquad a_j > 0,\ \sum_{j=1}^{m} \alpha_j = n$$

where α_j is the number of symbols of length t_j. Under these conditions the characteristic equation of the difference equation

$$N(T) = \sum_{k=1}^{m} \alpha_k N(T - t_k)$$

becomes

$$1 = \sum_{k=1}^{m} \alpha_k r^{-t_k}$$

The boundary condition can be derived from the equation:

$$N(t < t_1) = 0$$

The principal problem under investigation here is the determination of $\lim \frac{1}{T} N(T)$, when T is increased beyond any limit. For such infinitely large values of T, the function N(T) behaves as $A_i r_i^T$ where r_i is the largest positive root of the characteristic equation. Of course, it is not apparent that the characteristic equation does not possess a pair of complex conjugate roots r_j and r_{j+1} such that $A_j r_j^T + A_{j+1} r_{j+1}^T$ and not $A_i r_i^T$ is the dominant term.

The truth of the matter is that in both general cases cited before the root of the characteristic equation with largest magnitude is a positive one. The limit of N(T) for large T, thus approaches $A_i r_i^T$ without any concern that the contribution of the complex roots r_j and r_{j+1} may provide a dominant term.

While the validity of this statement has been unanimously recognized (for instance see Ref. 1,2,3,4,5),a complete proof may not be readily accessible in the literature.

In the following we proceed to give a proof for these statements in both cases. The suggested proof is not restricted to the case where all t_k's are positive integers. It will be shown that the positive real root of the characteristic equation is the root with the largest magnitude. This seems to be a stronger statement than those which have appeared earlier in the edited references. Furthermore, the method of proof outlined below may be of additional interest for solving analogous problems.

The characteristic equation

$$1 - \sum_{i=1}^{m} \alpha_i \, r^{-t_i} = 0$$

leads to an equation of the form

$$F(x) = x^{m'} - \beta_1 x^{m'-k_1} - \beta_2 x^{m'-k_2} - \ldots \beta_n x^{m'-k_n}$$

where $x = r^{t_o}$ and t_o is the greatest common divisor of t_i's. Now, the following main theorem can be stated.

Theorem: Consider the equation

$$F(x) = x^{m'} - \beta_1 x^{m'-k_2} - \ldots - \beta_n x^{m'-k_n} = 0$$

where $\beta_j \geq 0 \qquad j = 1, 2, \ldots, n \qquad n < m'$

and all β_j's vanish simultaneously.

The equation has a unique positive root $r_1 > 0$; the magnitude of any other root of this equation is not larger than r_1.

<u>Proof</u> The following proof is based on the work of A. M. Ostrowski (Ref. 6).

Let $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be the sequence of all positive coefficients ordered according to their indices, that is:

$$1 \leq k_1 < k_2 < \ldots < k_n \leq m'$$

There exist n integers $s_1, s_2, \ldots, s_n$ such that

$$s_1 k_1 + s_2 k_2 + \ldots + s_n k_n = 1$$

Now $F(x)$ can be rewritten as $\phi(x) \cdot x^{m'}$ where

$$\phi(x) = 1 - \left(\frac{\beta_1}{x^{k_1}} + \frac{\beta_2}{x^{k_2}} + \cdots\cdots + \frac{\beta_n}{x^{k_n}}\right) = 0$$

The function $\phi(x)$ is monotonically increasing in the range $0 < x < \infty$.

We note that $\phi(\infty) = +1$ and $\phi(0) = -\infty$. Therefore, $\phi(x)$ will vanish exactly for one positive value of x say $x = x_1$. (Note that $x_1 > 1$). Since, $\phi'(x_1) > 0$, it is concluded that the root x_1 cannot be a multiple root.

It is not apparent however, if the magnitude of this root is not smaller than the magnitude of any other root. To this end, let x_2 be any other root of the characteristic equation. Then

$$1 = \frac{\beta_1}{x_2^{k_1}} + \cdots\cdots + \frac{\beta_n}{x_2^{k_n}} \leq \frac{\beta_1}{|x_2|^{k_1}} + \cdots\cdots \frac{\beta_n}{|x_2|^{k_n}}$$

Thus $\phi(|x_2|) < 0$. Now it becomes clear that if $\phi(|x_2|) < 0$ then $|x_2| < x_1$. If $\phi(|x_2|) = 0$, then the following expression must be positive as it consists of factors which are all positive by virtue of the assumed relation

$$\left(\frac{\beta_1}{|x_2|^{k_1}}\right)^{s_1} \left(\frac{\beta_2}{|x_2|^{k_2}}\right)^{s_2} \cdots \left(\frac{\beta_n}{|x_2|^{k_n}}\right)^{s_n} = \frac{\beta_1^{s_1} \beta_2^{s_2} \cdots \beta_n^{s_n}}{|x_2|} > 0$$

Therefore $|x_2|$ must be also positive. This is in contradiction with the proven fact that the equation has exactly one positive root, unless for negative or complex roots having the same magnitude as x_1, that is $|x_2| = x_1$.

In the light of this theorem, it becomes clear that when T becomes exceedingly large, the dominant term of N(T) consists solely of terms containing roots whose magnitude are equal to the magnitude of the largest root. Of course, these roots may be complex conjugate pairs or even negative real. Due to the realness of N(T), a pair of conjugate roots contribute a real term of the form

$$A_k r_k^T + A_k^* r_k^{*T} = A'_k |r_k|^T$$

Thus, the dominant term will be of the form $A_0 |r_k|^T$ where A_0 may contain a contribution of negative or complex conjugate roots. But at any rate A_0 cannot be negative. This leads to Shannon's well-known result that:

$$C = \lim_{T \to \infty} \frac{1}{T} \log N(T) = \log |r_k|$$

Part II

A NOTE ON A SAMPLING THEOREM

The impact of the so-called sampling theorem is quite well-known to most communication engineers. In its simplest form, the theroem states that: a signal whose Fourier integral transform is limited to the frequency band of $\pm\ \omega_0$ cps, is completely determined by its sampled values at the intervals π/ω_0 seconds apart starting from the origin.

The sampling theorem has been extensively used in communication theory in a variety of forms. For example, we know that the sampling intervals may not necessarily be chosen with equal length. The knowledge of the value of the signal at an instant may be traded for an equivalent information about the derivative of the signal. Furthermore, the theorem may be extended from a one-dimensional signal space to a multidimensional signal space.[1]

While the context of this theorem has been mathematically known for over half a century, its introduction in communication engineering is due to C.E. Shannon. The theorem, however, was previously known or occasionally applied by other pioneers such as Nyquist, Kumfmüller and D. Gabor. It is due to the efforts of these and other communication scientists that today the sampling theorem has become a standard tool of research in communication engineering.

The object of the present paper is to suggest another mathematically known theorem which should prove to be, potentially, a rich source of information in problems of communication. The theorem was discovered in about 1912 by the famous mathematician S. Bernstein while giving another proof to the Weierstrass's theorem.

It is recalled that Weierstrass's theorem is one of the basic theorems of mathematics in the field of the theory of functions of real variables. This theorem states that any continuous function $f(t)$ of a real variable t in the closed interval $[a,b]$ can be approximated by a polynomial $P_n(t)$ of a specified degree n, such that for all other polynomials of degree n we have:

$$\max_{a \leq t \leq b} |f(t) - P_n(t)| \leq \max_{a \leq t \leq b} |f(t) - Q_n(t)| \tag{1}$$

Furthermore, for any arbitrary small $\epsilon > 0$, there exists a polynomial $P_n(t)$ (where n depends on ϵ) such that in the interval $[a,b]$ we have[2]

$$|f(t) - P_n(t)| \leq \epsilon \tag{2}$$

We may note, in passing, that there is a resemblance between Weierstrass's theorem and Shannon's fundamental theorem of information theory. The latter theorem, in its most familiar form, states that for a given communication system and specified error probability ϵ, there is an encoding-decoding scheme which leads to a transmission of information with a rate as close to the ideal rate as desired and with an error probabiltiy not exceeding ϵ (for a complete statement see Ref. 1 Chapter 12). Parenthetically, one may say that Shannon's fundamental theorem, in a way, is the counterpart of Weierstrass's basic theorem for communication sciences. Both these theorems suggest the possibility of approaching certain ideal behaviors with prespecified errors.

Without loss of generality, we assume that the continuous function $f(x)$ is defined in the interval $[0,1]$. Of course, any other real interval can

can be transformed into this interval by a simple transformation. The desired sampling theorem can be stated in terms of Bernstein's polynomials associated with sampling of f(x) at specified intervals of length 1/n where $n > 0$ is any specified positive integer and

$$B_n(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1 - t)^{n-k} f(\tfrac{k}{n}) \tag{3}$$

A graphical interpretation of the theorem for $n = 8$ is given in Fig. 1.

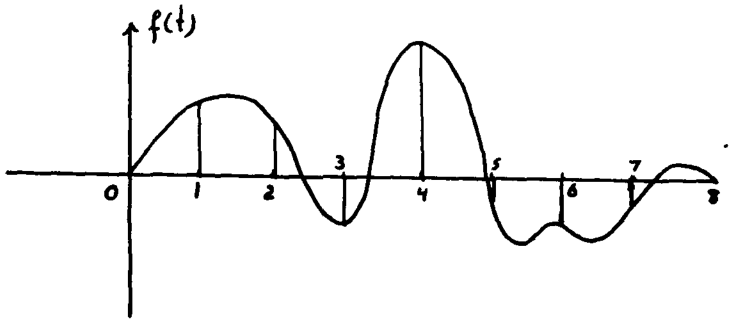

Fig. 1

The given function is sampled at integral points 0 to n. Each ordinate $f(\frac{k}{n})$ is computed and incorporated in the sampling kernel of Eq. (3); and summed up in order to lead to $B_n(t)$. Now according to the theorem of Weierstrass the limit of $B_n(t)$ as $n \to \infty$ approaches f(x) uniformly in the above interval, that is,

$$\lim_{n \to \infty} B_n(t) = f(x)$$

A standard proof for this theorem can be found in [3]. The following proof is due to S. Bernstein.

Let

$$P_k = \binom{n}{k} t^k(1 - t)^{n-k} \tag{4}$$

and note that

$$\sum_{k=0}^{n} P_k = 1 \tag{5}$$

Next we compute:

$$T = \sum_{k=0}^{n} (k - nt)^2 P_k = \sum_{k=0}^{n} [k(k-1) - (2nt - 1)k + n^2t^2] P_k \tag{6}$$

We may consider X to be a random variable with binomial distribution (see Ref. 1, page 234), that is,

$$P\{X = k\} = P_k = \binom{n}{k} t^k(1 - t)^{n-k} \tag{7}$$

Thus,

$$\bar{X} = \sum_{k=0}^{n} k \binom{n}{k} t^k(1 - t)^{n-k} = nt \tag{8}$$

$$\overline{X^2} = nt [(n - 1)t + 1] \tag{9}$$

$$\sigma^2 = \overline{X^2} - \bar{X}^2 = nt (1 - t) \tag{10}$$

$$\sum_{k=0}^{n} k(k - 1)P_k = \overline{X^2} - \bar{X} = n(n - 1)t^2 \tag{11}$$

Therefore

$$T = n(n - 1)t^2 - (2nt - 1)nt + n^2t^2 = nt(1 - t) \tag{12}$$

Our immediate problem is to evaluate the difference:

$$|f(t) - B_n(t)| = \left|\sum_{k=0}^{n} f(t)\, P_k - \sum_{k=0}^{n} - f(\tfrac{k}{n})\, P_k\right| \tag{13}$$

$$\leq \sum_{|\frac{k}{n} - t| < \delta}^{n} |f(t) - f(\tfrac{k}{n})|\, P_k \tag{14}$$

$$+ \sum_{|\frac{k}{n} - t| \geq \delta}^{n} |f(t) - f(\tfrac{k}{n})|\, P_k$$

On account of the continuity assumption for every point x' such that $|x - x'| < \delta$, we have $|f(x) - f(x')| < \epsilon$. Therefore, the total deviation in the interval $|\frac{k}{n} - t| < \delta$ cannot exceed ϵ.

A Chebyshev's type inequality can be established for the term to the right of the above inequality, that is,

$$\sum_{|\frac{k}{n} - t| > \delta}^{n} P_k \leq \frac{1}{\delta^2} \sum_{|\frac{k}{n} - t| \geq \delta}^{n} (\tfrac{k}{n} - t)^2\, P_k \leq \frac{1}{n^2\delta^2}\, T = \frac{t(1 - t)}{n\delta^2} \tag{15}$$

But in the range $[0,1]$, we have $t(1 - t) \leq \frac{1}{4}$, thus,

$$\sum_{\left|\frac{k}{n} - t\right| \geq \delta}^{n} P_k \leq \frac{1}{4n\delta^2} \tag{16}$$

From the above one concludes that

$$|f(t) - B_n(t)| \leq \epsilon + \frac{2M}{4n\delta^2} \tag{17}$$

where M is the largest value of f(t) in the interval [0,1].

When n is made arbitrarily large, for any specified ϵ and M we will have

$$\lim_{n \to \infty} |f(t) - B_n(t)| \leq 2\epsilon \tag{18}$$

This proves the uniform convergence of the sequence as stated in the theorem of Weierstrass.

The following remarks seem to be of some pertinent interest.

1. The proof of the above theorem suggests the inequality

$$|f(t) - B_n(t)| \leq \epsilon_n + \frac{M}{2\sqrt{n}} \tag{19}$$

where

$$\epsilon_n = \operatorname{Max} |f(t) - f(\tfrac{k}{n})| \tag{20}$$

$$\left|\frac{k}{n} - t\right| \leq \frac{1}{\sqrt[4]{n}}$$

It is also possible to find sharper inequalities.

2. It appears that the now well-known methods employed by A. Feinstein,

C.E. Shannon and J. Wolfowitz for proving the fundamental channel capacity theorem of information theory are in a certain sense similar to those employed for proving the theorem of Weierstrass. In both instances, certain inequalities are developed between specified ϵ, δ and a desired n- the inequalities give quantitative measures for estimating how fast the employed method approaches the ideal behavior. As an immediate example consider and compare the following theorems due to Popoviciu[5] and Wolfowitz respectively. a) If f(t) is a continuous function in [0,1] and $\omega(\delta)$ the modulus of continuity of f(t), then there exists a Bernstein polynomial $B_n(t)$, such that for sufficiently large, n,

$$|f(t) - B_n(t)| \leq k\, \omega(1/\sqrt{n}) \tag{21}$$

where k is a constant.

b) There exists a code book containing at least N words of length n such that for any arbitrary specified probability of error $\lambda (0 < \lambda \leq 1)$; there is a decoding scheme with a uniform error probability λ and $N \geq \exp\,(nC - k\sqrt{n})$ where k is a constant depending on λ but not on channel capacity C (see Ref. 1, page 421).

One of the communication problems of interest that appears yet unsolved is the following problem originated in the field of reliability.

Let S be a system of interconnected elements which is supposed to function between two node terminals A and B. We assume a certain 'flow' say flow of electric current between A and B. A question which was asked and adequately answered in the literature [see for instance 6 and 7] is the determination of the reliability function h(p) for that system knowing that every element has a reliability of operation $0 \leq p \leq 1$. The following properties of $h(p)$ are well-known.

1) $h(0) = 0$

2) $h(1) = 1$

3) $0 \leq h(p) \leq 1$

4) $h(p)$ is a monotonically increasing function in the closed interval $[0,1]$.

5) The equation $h(p)$ can have at the most one real root in the interval $0 \leq p \leq 1$

Questions dealing with realizability are generally quite complex. The field of probabilistic networks makes no exception to this statement. Given a reliability function $h(p)$ real for real p and satisfying all the above five conditions is it always possible to find a system S such that its r eliability function approaches $h(p)$ as close as desired? While the writer knows of no formal proof for this theorem, a positive answer to the question is conjectured here.

The aforementioned theorem guarantees that the specified $h(p)$ function can be synthesized in terms of Bernstein polynomials with sufficiently large n,

$$B_n(p)\sum_{k=0}^{n} \binom{n}{k} p^k(1 - p)^{n-k} f(k/n) \tag{21}$$

in the sense described earlier. There are two main difficulties before an existence realizability proof can be even outlined: In the first place, for a realizable discrete system, $\binom{n}{k} f(k/n)$ must be an integer for all integral values of k between 0 and n. In the second place, even assuming that the letter numbers have all integrals values it is not clear at all whether all integral coefficients thus derived can collectively be presented by a linear graph. Of course, for any specified graph, there is a definite relationship between number of paths of K elements long (of length k) between nodes A and B.

The first of these difficulties can be removed, for instance, in the light of the work of Russian mathematicians M.I. Chlodovsky [8] and L.V. Kantorowich [9]. Chlodovsky has proved that the Weierstrass theorem can be alternatively approached by Bernstein-like polynomials with integral coefficients, that is

$$C_n(t) = \sum_{k=1}^{n-1} \left[\binom{n}{k} f\left(\frac{k}{n}\right)\right] t^k(1 - t)^{n-k} \tag{22}$$

The bracketed numbers [ ] stands for the largest integer written in the bracket. Therefore

$$\lim_{n \to \infty} |h(p) - C_n(p)| \longrightarrow 0 \tag{23}$$

The second difficulty appears to be a more serious one at this time. While no specific proof is presently available to support or to contradict the aforementioned conjecture, it is hoped that we have called attention to an important problem and the existence of a suitable tool for handling problems of this sort. More investigation seems to be desirable in this area.

A NOTE ON RELIABILITY FUNCTIONS

Part III

INTRODUCTION

This communication is based on a talk delivered before the 2^{nd} International Congress on Cybernetics.* The object of the talk was two fold. The first objective was to present a discussion before the Congress of a significant contribution recently made to the Theory of Probabilistic Networks in the United States [2]. The second objective was to supplement the work of reference [2] with some observations and theorems which might prove of interest in the future development of the subject.

Subsequent to some original suggestions of the late J. Von Neumann, in a series of lectures given at California Institute of Technology in 1952 [7], Drs. E.F. Moore and C.E. Shannon established the basis for a new area of research in the field of Communication Engineering. In fact, their paper "Reliable circuits using less reliable relays" established the basic rules for studying probabilistic networks. The work of reference [2] may in the future be considered a parallel to the instigation of developments in the field of switching circuits brought about by reference [6].

The present note does not include a summary of Shannon-Moore's fundamental work, as presented in the original talk to the Congress. This note is mainly confined to the second objective of the talk. In section 2, we shall discuss the concept of h(p) functions as introduced by Von Neumann, Moore and Shannon. Section 3 offers the writer's proof of the Shannon-Moore

*September 3-10, 1958, Namur, Belgium.

expansion theorem. An extension of the latter theorem is given in section 4. Section 5 presents an additional theorem on necessary conditions of realizability, and emphasizes an unsolved central problem of this field. A tie between Reliability Functions and Positive Real Functions is introduced. Certain formulae concerning the interrelationships of the coefficients of Reliability Functions for series, parallel and composition of networks have been developed along with some bounds for these coefficients.

Reliability Functions

We shall consider a connected network N with n branches and two terminal nodes A and B. For instance N may be the graph of an equipment with n components. The node A receives what is called the input and the node B transmits the output of the system. In a physical sense we tacitly assume a "flow" of some sort in the network, that is, we postulate that the operation of the system requires the functioning of a set of connected branches (components) between nodes A and B. Our major problem is to study the reliability of such a system in terms of the reliability of its components. In other words if the probability of the operation of the branch k (assumed to be independent of the probability of operation of any other branch) is denoted by p_k, find the probability of the operation of the system:

$$h(p_1, p_2, \ldots p_n) \tag{1}$$

Evidently for a finite lumped network the reliability function (1) is a polynomial of degree n with integer coefficients. Furthermore:

$$h(0,0,0, \ldots 0) = 0 \tag{2}$$

$$h(1,1,1, \ldots 1) = 1 \tag{3}$$

$$0 \leq h \leq 1 \tag{4}$$

when all p_k's remain between zero and one.

The principal theoretical problems in this field seem to encompass a thorough study of the function h(p), that is, an analysis and a synthesis of Reliability Functions. The fact that we are dealing with the probability of the operation of each component and the graph of the system suggests that the above problem can be formulated as a problem of linear graphs where a probability weighting function is associated with each branch. In other words, from a theoretical stand point, the reliability study of the servomechanism incorporated in an automatic milling machine and the study of the reliability of an electronic computer in a missile are basically the same. In both cases, the nature of the interconnection of the components (the graph) and the weighting function associated with each branch completely determine the reliability of the system. The same mathematical tools and procedures apply in both cases. The observation is rather important since it might assist in reducing the enormous technical literature of the subject to a handful of fundamental procedures .

In the present paper, following Shannon and Moore, we generally confine ourselves to the simplest case where:

$$p_k = p \qquad\qquad k = 1, 2, \ldots n,$$

Let E_k be the event that the network operates between nodes A and B when a set consisting of k connected branches operate properly and the n-k remaining branches fail to function. The event E of the operation of the system between A and B is given by:

$$E = \sum_{k=1}^{n} E_k \tag{5}$$

Of course the event E_k may occur in a number of A_k distinct but equivalent ways. For example, in Figure 1, the event E_2 can occur in three distinct ways:

(1-2, 3-2, and 4-5)

Thus, $A_2 = 3$.

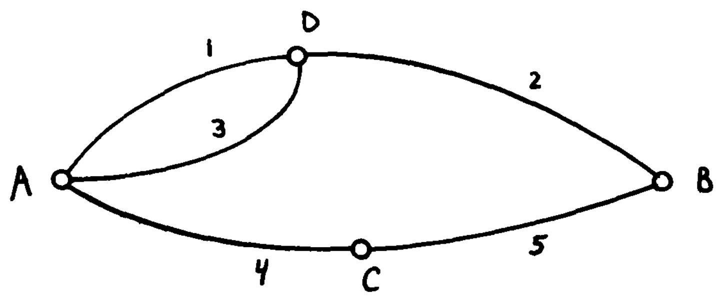

FIG. I

The probability of the successful operation of the network under the event E_k is:

$$P_r(E_k) = A_k p(1 - p^k)^{n-k} \tag{6}$$

Since the E_k's are mutually exclusive subsets of E, then:

$$h(p) = P_r(E) = \sum_{k=1}^{n} P_r(E_k) = \sum_{k=1}^{n} A_k p(1 - p^k)^{n-k} \tag{7}$$

This is the formula suggested in reference [2]. The following properties of h(p) are known:

1. $h(0) = 0$ (8)
2. $h(1) = 1$ (9)
3. $0 \leq h(p) \leq 1$ (10)
4. h(p) is a monotonically increasing function in the interval $0 \leq p \leq 1$ (11)
5. $h(p) = p$ can have at the most one real root in the interval $0 \leq p \leq 1$ (12)

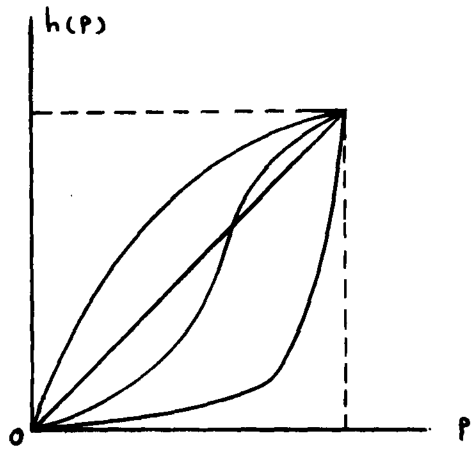

FIG. 2

An Expansion Theorem

An interesting expansion theorem has been employed by Shannon-Moore [2]. The theorem permits the derivation of h(p) of a complex network containing n branches from the reliability functions of two associated networks, each having n-1 branches between the same input-output terminals. Along with the above set-theoretic explanation the following proof for the Shannon-Moore theorem is suggested. We shall divide the set of events E_k's in two categories, those whose occurrence requires the inclusion of the following subevent: "The events requiring the functioning of a particular branch b" and "those events which are not contingent on the functioning of the branch b". These two subsets of E are respectively denoted by X and Y. Evidently if the occurrence of a particular event E_j requires the functioning of the branch b, then $E_j \subset X$. If E_j does not require the functioning of the branch b, then it will require the failure of that branch, thus $E_j \subset Y$. Each member of the set X is a set product containing the event of the operation of the branch b. This event will be denoted by Q. Therefore:

$$X = QY_0 \tag{13}$$

$$Y = Q'Y_0 \tag{14}$$

$$E = X + Y = QX_0 + Q'Y_0 \tag{15}$$

The probabilities associated with the two mutually exclusive sets of equation (15) satisfy:

$$P_r(E) = P_r(QX_0) + P_r(Q'Y_0) \tag{16}$$

$$P_r(E) = P_r(Q)P_r(X_0) + P_r(Q')P_r(Y_0) \tag{17}$$

By letting:

$$P_r(X_0) = f(p) \tag{18}$$

$$P_r(Y_0) = g(p) \tag{19}$$

One obtains:

$$h(p) = pf(p) + (1 - p)g(p) \tag{20}$$

The network interpretation of this equation can be given as follows. Any event $x_0 \subset X_0$ implies that the network N operates between the nodes A and B only via branch b. That is if the two nodes across the branch b are shorted we obtain a graph with reliability function f(p) between nodes A and B. Similarly any $y_0 \subset Y_0$ implies that the network N operates between nodes A and B while the branch b is removed. That is if the branch b is removed from N, there remains a connected network with a reliability function h(p) between nodes A and B. Based on this proof, the Shannon-Moore theorem can be summarized as below:

---Short circuit a particular branch b, compute f(p);

---Remove the branch b compute g(p);

---Obtain h(p) from equation (20).

The equation (20) gives an expansion rule for the network reliability function in terms of the reliability of one of its branches and the remainder network. This theorem is perhaps similar to the familiar Thévenin theorem of electrical circuits.

A more analogous situation can be brought in focus by considering a linear transformation of an impedance load z by an ordinary two-port. The input impedance of the network of Figure 3a is:

$$Z(s) = \frac{Az + B}{Cz + D} \tag{21}$$

where (A,B,C,D) are the transmission function associated with the two-port and s the familiar complex frequency.

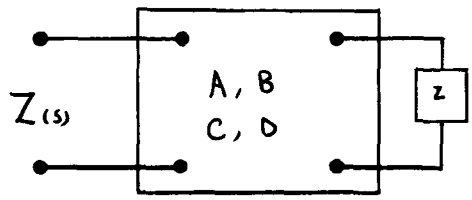

Figure 3a

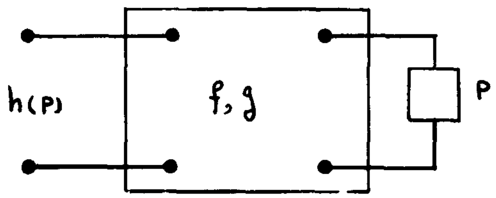

Figure 3b

In this classical network approach, we are essentially expanding a driving-point impedance function Z in terms of the impedance of a particular branch and systems functions associated with the remainder of the network evaluated at the terminals of the same branch. Similarly the Shannon-Moore theorem gives an expansion formula for h(p) in terms of the branch under consideration and the reliability functions associated with the terminals of the said branch. As an example of the application of this theorem, one may compute the function h(p) associated with AB of the bridge network of Figure 4.

$$f(p) = 4p^2(1 - p)^2 + 4p^3(1 - p) + p^4 = p^4 - 4p^3 + 4p^2$$

$$g(p) = 2p^2(1 - p)^2 + 4p^3(1 - p) + p^4 = -p^4 + 2p^2$$

$$h(p) = pf + (1 - p)g = 2p^2 + 2p^3 - 5p^4 + 2p^5$$

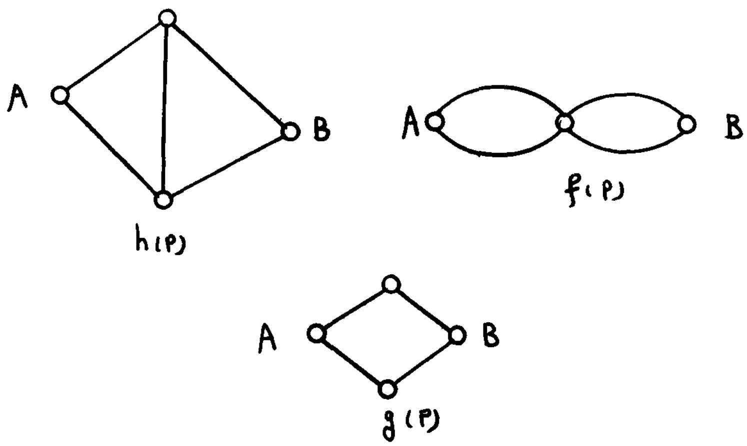

Figure 4

An Extension of Shannon-Moore Theorem

The above expansion theorem appears to be a source for generation of other interesting theorems. For instance if one wants to expand the reliability function h(p) associated with a network N in terms of the branches 1 and 2, one may first write:

$$h = f_1 p_1 + (1 - p_1) g_1 \tag{22}$$

Now f_1 and g_1 in turn can be expressed in terms of expansions of f_1 and g_1 about a branch with reliability p_2; we find:

$$f_1 = f_{21} p_2 + (1 - p_2) g_{21} \tag{23}$$

$$g_1 = f_{22} p_2 + (1 - p_2) g_{22} \tag{24}$$

Finally for the case $p_1 = p_2 = p$

$$h = f_{21}p^2 + p(1 - p)(f_{22} - g_{21}) + (1 - p)^2 g_{22} \tag{25}$$

The interpretation of this formula is straight forward:

f_{21} Reliability function when branch 1 and 2 are shorted;

f_{22} Reliability function when branch 1 removed and 2 shorted;

g_{21} Reliability function when branch 1 shorted and 2 removed;

g_{22} Reliability function when branch 1 and 2 are removed.

This result can be graphically interpreted by the tree diagram of Figure 5.

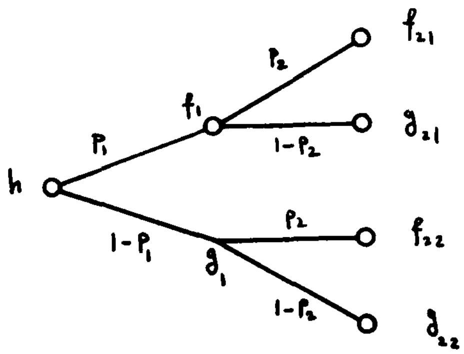

Figure 5

The over all probabilistic length of the trees to the right of each node gives a pertinent expansion rule. For example for the node h we have:

$$\begin{aligned} h = p_1f_1 + (1 - p_1)g_1 = p_1[p_2f_{21} + (1 - p_2)g_{21}] \\ + (1 - p_1) [p_2f_{22} + (1 - p_2)g_{22}] \end{aligned} \tag{26}$$

This procedure can be extended in a direct fashion to a general expansion theorem for obtaining the reliability function in terms of reliability functions associated with closing and removal of any combination of a number of specified elements.

An equivalent analytical formulation can be obtained from the following relations:

$$h = [f_1, g_1] \begin{bmatrix} p_1 \\ 1 - p_1 \end{bmatrix} \tag{27}$$

$$[f_1, g_1]^t = \begin{bmatrix} f_{21} & g_{21} \\ f_{22} & g_{22} \end{bmatrix} \begin{bmatrix} p_2 \\ 1 - p_2 \end{bmatrix} \tag{28}$$

$$h = \begin{bmatrix} p_2 \\ 1 - p_2 \end{bmatrix}^t \cdot \begin{bmatrix} f_{21} & g_{21} \\ f_{22} & g_{22} \end{bmatrix}^t \begin{bmatrix} p_1 \\ 1 - p_1 \end{bmatrix} = [p_2, 1 - p_2] \begin{bmatrix} f_{21} & f_{22} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ 1 - p_1 \end{bmatrix} \tag{29}$$

Other equivalent formulation and generalization of the relation (29) are possible. This will not be undertaken at present.

Realizability

From an academic point of view, perhaps the most needed theory in the field of probabilistic networks is a theory describing the necessary and sufficient conditions for a function $h(p)$ to be the reliability function of a discrete network. To mention an analogy, the significant development of the field of network synthesis is primarily due to the introduction of the concept of Positive Real Functions, or what is commonly known driving-point impedance functions. Given a one-port linear network we associate with it a driving-point impedance function $Z(s)$. The function $Z(s)$ is analytic in the right half-plane and such that:

$\text{Re } Z(s) \geq 0 \quad \text{for Re } s \geq 0$

$Z(s)$ is real on the real axis.

Conversely, given a function $Z(s)$ satisfying the above conditions, we may find a network N having $Z(s)$ for its driving-point impedance function. The central question of realizability of $h(p)$ functions is admittedly a difficult one. A method for describing this class of functions and testing if a function belongs to that class has not been suggested in the literature. The following is hoped to be a step in that direction.

A tentative sketch of a fundamental theorem in the field can be made by exploiting the strong constraint of the monotonically increasing character of the reliability function. We shall first transform the interval $(0,1)$ onto the interval $(0,2\pi)$. Then we shall translate the constraint of monotonically increasing into a condition on the coefficients of the Fourier series expansion of the new function. This will provide a link between the realizability problem and the theory of Harmonic Functions. Finally we will attempt to make a connection between the realizability of probabilistic networks and the realizability of deterministic networks i.e. a tie between $h(p)$ and Positive Real Functions.

Let:

$$p = \frac{x}{2\pi} \tag{30}$$

This transformation transforms the interval $0 \leq p \leq 1$ onto $0 \leq x \leq 2\pi$. We may now consider a periodic function $\phi(x)$ which coincides with the function $h\left(\frac{x}{2\pi}\right)$ in the interval $0 \leq x < 2\pi$. This function can be expanded in a Fourier series:

$$\phi(s) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (C_k \epsilon^{jkx} + \bar{C}_k \epsilon^{-jkx}) \tag{31}$$

where:

$$\begin{aligned} C_0 &= -ja_0 \\ C_k &= b_k - ja_k \\ \bar{C}_k &= b_k + ja_k \\ C_{-1} &= -b_1 - ja_1 \end{aligned} \tag{32}$$

Now the following central theorem can be established.

1. Theorem.

A set of necessary conditions for a given function $h(p)$, real on the real axis, to be a permissible reliability function are:

1. $h(0) = 0 \quad , \quad h(1) = 1$
2. $0 < h(p) < 1$ for $0 < p < 1$
3. $h(p) = p$ has at the most one positive real root in:

$$0 < p < 1$$

4. $$F(z) = \sum_{n=0}^{\infty} \left(nC_n - \frac{n+1}{2} C_{n+1} - \frac{n-1}{2} C_{n-1}\right) z^n \tag{33}$$

has a positive real part in the unit circle.

2. Proof.

Conditions (1) and (2) are self evident. Condition (3) has been obtained by Shannon-Moore in reference [2]. An elaborate procedure is required to show that condition (4) constitutes the contribution of this note. A detailed account would require much time and space. However a summary of the proof will be given below. The following theorem has been proved by Herglotz [1][8][9].

Every function regular in the interior of the unit-circle and with real part non-negative in the unit-circle can be represented as:

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\epsilon^{ja} + \lambda}{\epsilon^{ja} - \lambda} d\,\psi(\alpha) + K \tag{34}$$

where $\psi(\alpha)$ is a non decreasing bounded function, K a pure imaginary constant and the integral taken in the Stieltges sense.

It is a matter of using appropriate transformations in order to derive a theorem equivalent to Herglotz theorem for functions analytic in the right half-plane and with positive real part in the right half-plane.

Applying the transformation:

$$y = -\cot\frac{x}{2} \tag{35}$$

one finds:

$$\cos x = \frac{y^2 - 1}{y^2 + 1} \tag{36}$$

$$\sin x = \frac{-2y}{y^2 + 1} \tag{37}$$

$$e^{jx} = \frac{y - j}{y + j} \tag{38}$$

$$e^{-jx} = \frac{y + j}{y - j} \tag{39}$$

$$f(y) = \frac{a_0}{2} + \sum_{k=1}^{\infty} C_k\left(\frac{y - j}{y + j}\right)^k + \overline{C}_k\left(\frac{y + j}{y - j}\right)^k \tag{40}$$

Subsequent to the application of a number of transformations primarily for applying the Herglotz theorem, one finds that the function:

$$\sum_{n=0}^{\infty} \left(nC_n - \frac{n+1}{2} C_{n+1} - \frac{n-1}{2} C_{n-1}\right)z^n \tag{41}$$

has positive real part in the unit circle $|z| < 1$. This result is based upon the derivation of the function d $\phi_1(x)$ and inserting it in the appropriate equivalent form of Herglotz integral and then obtaining the residues. This shows the necessity of the condition.

The sufficiency of the above condition for monotonically increasingness of h(p) can be also proved, following a number of involved procedures, from the converse property of Herglotz theorem. That is, the condition (4) is sufficient in order to lead to a monotonically increasing h(p). It does not formally follow from this that the obtained function h(p) will correspond to a realizable network.

While the above theorem is of no practical significance it has certain interesting impacts. In fact, with a further constraint of $F(z) = \tilde{F}(\bar{z})$ the function F(z) can easily lead to a function $Z(s) = F(z = \frac{1 - s}{1 + s})$, which is a Positive Real Function. Thus the suggested theorem has a curious property of relating the reliability functions of probabilistic networks to the impedance functions of the familiar circuit theory.

A number of relationships between coefficients A_k's of the reliability function of the series, parallel or composition of networks have been obtained by F. Moskowitz, S. Jutila, and this writer. These are given in references [3] and [4]. Also certain upper bounds for coefficients A_k and some inequalities for reliability functions have been obtained. This will be presented at a later date.

Part IV

Nearly Gaussian Channels

We have formulated and studied the following problem for transmission of information in "nearly normal channels".

Let F(x) be the C.D.F. of a "nearly normal distribution", that is

$$\left| F(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \, dt \right| < \epsilon$$

$$0 < \epsilon < 1 \qquad -\infty < x < \infty$$

The output of this source is transmitted through a nearly normal channel in the presence of the same type of additive noise. The main problem is to obtain some bounds on the entropy of nearly normal sources and channels, and then capacity.

This problem is far too complex and requires more time for future investigation. So far, we have employed methods of inquiry similar to Gram-Charlier Series of orthogonal expansion. Also we have used theorems describing certain distances between characteristic functions of two distributions. For instance let f(x) and g(x) be two distibution functions; F(t) and G(t) their respective characteristic functions such that F(t) coincides with G(t) on the interval $|t| < L$. Then one can prove that the deviation of f(x) and g(x) must satisfy an interesting inequality.

$$\int_{-\infty}^{\infty} |f(x) - g(x)| \, dx \leq \frac{\pi}{L}$$

A similar bound for the entropies has been under investigation.

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